RINGS OF INTEGER-VALUED CONTINUOUS FUNCTIONS

BY

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Introduction. The purpose of this paper is to study the ring $C(X, Z)$ of all integer-valued continuous functions on a topological space X. Our subject is similar in many ways to the ring $C(X)$ of all real-valued continuous functions on X. It is not surprising therefore that the development of the paper closely follows the theory of $C(X)$.

During the past twenty years extensive work has been done on the ring $C(X)$. The pioneer papers in the subject are [8] for compact X and [3] for arbitrary X. A significant part of this work has recently been summarized in the book [2]. Concerning the ring $C(X, Z)$, very little has been written. This is natural, since $C(X, Z)$ is less important in problems of topology and analysis than $C(X)$. Nevertheless, for some problems of topology, analysis and algebra, $C(X, Z)$ is a useful tool. Moreover, a comparison of the theories of $C(X)$ and $C(X, Z)$ should illuminate those aspects of the theory of $C(X)$ which derive from the special properties of the field of real numbers. For these reasons it seems worthwhile to devote some attention to $C(X, Z)$.

The paper is divided into six sections. The first of these treats topological questions. An analogue of the Stone-Čech compactification is developed and studied. In §2, the ideals in $C(X, Z)$ are related to the filters in a certain lattice of sets. The correspondence is similar to that which exists between the ideals of $C(X)$ and the filters in the lattice of zero sets of continuous functions on X. This theory provides a characterization of those ideals of $C(X, Z)$ which are intersections of maximal ideals. §3 is concerned with the space of maximal ideals in $C(X, Z)$. In §4, some existence theorems for maximal ideals are proved. The residue class fields of $C(X, Z)$ modulo maximal ideals are studied in the last two sections. It turns out that those of prime characteristic are trivial: the integers modulo the characteristic. The residue class fields of characteristic zero are distinctly nontrivial. In §5, the cardinality of such fields is investigated. The main result is that they are always uncountable. In §6, the algebraic properties of the zero characteristic residue class fields are examined. It is shown for example that these fields are always quasi-algebraically closed.

As we noted above, very little has been published concerning the ring $C(X, Z)$. Nevertheless, a considerable number of "folk theorems" exist in the subject. One of our objectives in writing this paper is to get these results

into general circulation as concisely and efficiently as possible. Use is made of many people's ideas. H. H. Corson, Leonard Gillman and J. R. Isbell deserve special mention. But above all the paper owes its existence to Edwin Hewitt, who first stimulated the author's interest in $C(X, Z)$. Several theorems presented here are direct consequences of extended discussions with Hewitt. They are truly the products of joint research. For this generous help, the author extends his sincerest thanks to Professor Hewitt.

Our notation is largely taken from (or patterned after) [2]. As usual, Z represents the ring of integers and Q denotes the field of rationals. For any $n \in Z$, let Z_n be the residue class ring of Z modulo the ideal generated by n. Following [2], if $n \in Z$, the symbol n represents the constant function (on the space under consideration) whose value at each point is n. The set of all rational primes is denoted by Π. The cardinal number of a set S is designated $|S|$. The symbols for set and ring operations are standard. We use $\prod_i R_i$ to represent the direct product of the family of rings $\{R_i\}$. Finally, $C^*(X, Z)$ will denote the ring of all *bounded*, integer-valued, continuous functions on X.

1. Topological considerations.

1.1. Let X be an arbitrary topological space. By a *partition* of X, we will mean a countably infinite collection $\{V_i\}_{i=1}^{\infty}$ of open subsets of X such that $V_i \cap V_j = \varnothing$ if $i \neq j$ and $\cup_i V_i = X$. The partition is called *infinite* if infinitely many V_i are nonempty. Otherwise it is called *finite*. It is clear that each V_i in a partition is both open and closed. In fact, the union of any subset of a partition is open-and-closed. The Boolean algebra of all open-and-closed subsets of X will be designated by $\mathfrak{B}(X)$, or simply $\mathfrak{B}$ where reference to the space X is unnecessary.

1.2. LEMMA. *Let $\{V_i\}$ be a partition of X. Suppose that for each i, $f_i \in C(V_i, Z)$. Define $f: X \to Z$ by $f(x) = f_i(x)$ for $x \in V_i$. Then $f \in C(X, Z)$.*

An important case of this lemma is where each f_i is constant on V_i.

1.3. Along with the usual ring and lattice operations, it is possible to form quotients, remainders and greatest common divisors in $C(X, Z)$ and $C^*(X, Z)$.

LEMMA 1.3.1. *Let $f, g \in C(X, Z)$. Then functions d and r exist in $C(X, Z)$ such that $f = dg + r$ and $|r(x)| < |g(x)|$ if $g(x) \neq 0$. Moreover, d and r can be chosen so that $|d| \leq |f|$ and $|r| \leq |f|$. In particular, if $f \in C^*(X, Z)$, then it can be assumed that d and r are also in $C^*(X, Z)$.*

LEMMA 1.3.2. *Let $f, g \in C(X, Z)$. Let $d: X \to Z$ be the non-negative function such that $d(x)$ is the greatest common divisor of $f(x)$ and $g(x)$ for all $x \in X$. Then $d \in C(X, Z)$. Moreover, there exist functions a and b in $C(X, Z)$ such that $d = af + bg$. If f and g belong to $C^*(X, Z)$, then $d \in C^*(X, Z)$ and it is possible to choose a and b in $C^*(X, Z)$ also.*

These lemmas follow easily from 1.2, using the fact that if $\{U_i\}$ and $\{V_j\}$ are partitions of X, then so is $\{U_i \cap V_j\}$.

We will denote the non-negative greatest common divisor of f and g by (f, g).

1.4. Let ϕ be a continuous mapping of X into Y. Let ϕ^* be the mapping of $C(Y, Z)$ into $C(X, Z)$ defined by

$$(\phi^* f)(x) = f(\phi x).$$

It is evident that ϕ^* is a ring homomorphism. We will call ϕ^* the *adjoint* of the mapping ϕ. Note that ϕ^* maps $C^*(Y, Z)$ into $C^*(X, Z)$.

LEMMA 1.4.1. *ϕ^* is one-to-one if and only if Y contains no nonempty open-and-closed set disjoint from $\phi(X)$.*

Proof. If U is open-and-closed in Y and $U \cap \phi(X) = \varnothing$, then $\phi^* \chi_U = 0$, where χ_U is the characteristic function of U. Conversely, if $f \neq 0$ in $C(Y, Z)$ and $\phi^* f = 0$, then $U = \{y \in Y \,|\, f(y) \neq 0\}$ is a nonempty open-and-closed set in Y which is disjoint from $\phi(X)$.

DEFINITION 1.4.2. If ϕ^* maps $C(Y, Z)$ onto $C(X, Z)$, then ϕ is called a C_Z-*injection*. If ϕ^* maps $C^*(Y, Z)$ onto $C^*(X, Z)$, then ϕ is called a C_Z^*-*injection*. If ϕ is also one-to-one and bicontinuous, then it is called a C_Z-*embedding* (respectively C_Z^*-embedding).

Evidently, a one-to-one continuous mapping ϕ of X into Y is a C_Z-embedding (C_Z^*-embedding) if and only if every $f \in C(\phi X, Z)$ (respectively, $C^*(\phi X, Z)$) can be extended to a continuous, integer-valued function on Y.

1.5. Let S be a subset of $C(X, Z)$. Define P_S to be the topological product of the discrete spaces $f(X)$, $f \in S$. Let ϕ_S be the mapping from X into P_S defined by mapping x into the point $(\cdots f(x) \cdots)_{f \in S}$. Obviously, ϕ_S is continuous. For each $f \in S$, let f_0 be the function mapping P_S into Z defined by $f_0(\cdots x_f \cdots) = x_f$. That is, f_0 is the projection of P_S onto the f-coordinate. Hence, f_0 is continuous. By definition, $f_0(\phi_S(x)) = f(x)$ for all $x \in X$. In other words, the adjoint of ϕ_S maps f_0 onto f. These observations yield the following results in the special cases $S = C(X, Z)$ and $S = C^*(X, Z)$.

THEOREM 1.5.1. *Any space X admits a C_Z-injection into a topological product of countable discrete spaces.*

THEOREM 1.5.2. *If X is any space, there is a continuous mapping of X into a compact, totally disconnected (Hausdorff) space δX such that the adjoint mapping is an isomorphism of $C^*(\delta X, Z)$ onto $C^*(X, Z)$.*

Proof. Let $S = C^*(X, Z)$ and define δX to be the closure of $\phi_S X$ in P_S. Since $f(X)$ is finite for each $f \in S$, the Tychonoff theorem implies that P_S is compact.

1.6. We will show that the space δX and the mapping of X into δX (obtained in 1.5.2) are unique. First note that the isomorphism of $C^*(\delta X, Z)$

onto $C^*(X, Z)$ carries idempotents onto idempotents. Therefore, the Boolean algebras $\mathfrak{B}(\delta X)$ and $\mathfrak{B}(X)$ are isomorphic. Since δX is compact and totally disconnected, it is homeomorphic to the Boolean space of $\mathfrak{B}(\delta X)$ (see [8]).

THEOREM 1.6.1. *Up to homeomorphism, there is one and only one compact, totally disconnected space δX such that $C^*(X, Z) \cong C^*(\delta X, Z)$, namely, the Boolean space of $\mathfrak{B}(X)$.*

THEOREM 1.6.2. *Suppose that Y_1 and Y_2 are compact, totally disconnected spaces and that ϕ_1 and ϕ_2 are continuous mappings of X into Y_1 and Y_2 such that ϕ_1^* is an isomorphism of $C^*(Y_1, Z)$ onto $C^*(X, Z)$ and ϕ_2^* is an isomorphism of $C^*(Y_2, Z)$ onto $C^*(X, Z)$. Then there is a homeomorphism ψ of Y_2 onto Y_1 such that $\phi_1 = \psi\phi_2$.*

Proof. Since $\phi_2^{*-1}\phi_1^*$ is an isomorphism of the Boolean algebra of idempotents in $C^*(Y_1, Z)$ on the Boolean algebra of idempotents in $C^*(Y_2, Z)$, there is a homeomorphism ψ of Y_2 onto Y_1 such that $\psi^* = \phi_2^{*-1}\phi_1^*$, at least on the idempotents (see [8]). Then $(\psi\phi_2)^* = \phi_1^*$ (on idempotents). Suppose $\psi\phi_2(x) \neq \phi_1(x)$. Choose an open-and-closed set U in Y_1 containing $\phi_1(x)$ but not $\psi\phi_2(x)$. Then $\phi_1^*\chi_U(x) = \chi_U(\phi_1(x)) = 1 \neq 0 = \chi_U(\psi\phi_2(x)) = (\psi\phi_2)^*\chi_U(x)$, a contradiction. Thus, $\psi\phi_2 = \phi_1$.

Because of these theorems, we are justified in introducing the special symbols δX for the space having the properties described in 1.5.2. Moreover, the injection of X into δX is sufficiently unique to be given a name. Let θ designate this mapping.

It follows easily from the definition of the adjoint homomorphism that if U is an open-and-closed set in δX, then

$$\theta^*\chi_U = \chi_{\theta^{-1}U}.$$

This identity, together with the fact that θ^* maps the idempotents of $C^*(X, Z)$ onto idempotents, yields

LEMMA 1.7.1. *If V is an open-and-closed set in X, there is an open-and-closed set U in δX such that $V = \theta^{-1}(U)$. Hence, $\theta(V)$ is open-and-closed in $\theta(X)$. Moreover, $\theta(X - V) = \theta(X) - \theta(V)$.*

COROLLARY 1.7.2. *The injection $\theta: X \to \delta X$ is one-to-one if and only if the open-and-closed sets of X separate points. It is an embedding if and only if X is a T_1-space and the open-and-closed sets form a basis for the topology.*

Two sets S and T in a space X are said to be *completely separated* if there is a real-valued, continuous function on X which takes the value zero on S and the value one on T.

THEOREM 1.7.3. *Let X be a completely regular space and let βX be the Stone-Čech compactification of X. Then βX is homeomorphic to δX if and only if every pair of completely separated sets in X are separated by an open-and-closed set.*

Proof. By 1.6.2, βX is homeomorphic to δX if and only if βX is totally disconnected. Moreover, for a compact Hausdorff space, total disconnectivity is equivalent to the condition that completely separated subsets are separated by an open-and-closed set. Finally, since every bounded, real valued, continuous function on X has a unique extension to βX, the condition that completely separated sets are separated by open-and-closed sets is satisfied in βX if and only if it is satisfied in X.

REMARK. An example of a metric space in which the open-and-closed sets separate points, but do not form a basis can be found in [2, 16L]. This same reference (see 16M) contains an example of a space X such that X is not totally disconnected, but X does have a basis of open-and-closed sets.

1.8. As in the case of $C(X)$, the structure of $C(X, Z)$ is relatively simple if X has the property that all of its integral-valued, continuous functions are bounded.

DEFINITION 1.8.1. A space X is called Z-pseudocompact (abbreviated ZPC) if $C(X, Z) = C^*(X, Z)$.

The notion of Z-pseudocompactness is an obvious analogue of pseudo-compactness introduced in [3]. Any pseudocompact space is ZPC. So is any connected space.

LEMMA 1.8.2. *The following conditions on a topological space X are equivalent.*

(a) *X admits an infinite partition (see 1.1).*

(b) *There is a continuous mapping of X onto the discrete space of positive integers.*

(c) *X is not Z-pseudocompact.*

Proof. (a) implies (b) by 1.2; (b) implies (c) obviously; (c) implies (a), since if $f \in C(X, Z) - C^*(X, Z)$, the set of all $V_n = \{x \in X \mid f(x) = n\}$ is an infinite partition of X.

1.9. We now examine the relation between ZPC spaces and pseudocompact spaces. Let $\theta: X \to \delta X$ be the canonical mapping defined in 1.6.

LEMMA 1.9.1. *θ^* maps $C(\theta(X), Z)$ isomorphically onto $C(X, Z)$.*

Proof. By 1.4.1, θ^* is one-to-one. Let $f \in C(X, Z)$. If $f(x) \neq f(y)$, then there exists an open-and-closed set U containing x but not y. Therefore $\theta x \neq \theta y$ by 1.7.1. Thus, there is a function $g: \theta(X) \to Z$ such that $g(\theta x) = f(x)$ for all $x \in X$. If $n \in Z$, then $g^{-1}(\{n\}) = \{\theta x \mid g(\theta x) = n\} = \{\theta x \mid f(x) = n\} = \theta(\{x \mid f(x) = n\})$ is open in $\theta(X)$ by 1.7.1. Therefore g is continuous and $f = \theta^* g$. Since f was arbitrary, θ^* is onto.

It is clear that if δX is identified with the Boolean space of all ultrafilters of $\mathfrak{B}(X)$, then $\theta(X)$ is the subspace consisting of all ultrafilters with a nonempty intersection. Thus, the topological structure of $\theta(X)$ is easily

determined from X. Note that $\theta(X)$ is a Hausdorff space in which the open-and-closed sets form a neighborhood basis. Such spaces play the same role in the theory of $C(X, Z)$ as the completely regular spaces play in the theory of $C(X)$.

LEMMA 1.9.3. *If the space X has a basis of open-and-closed sets, then X is Z-pseudocompact if and only if it is pseudocompact.*

Proof. Suppose f is an unbounded, real-valued function on X. Then there is a sequence $x_1, x_2, \cdots, x_i, \cdots$ of points in X, and a sequence $r_1, r_2, \cdots, r_i, \cdots$ of real numbers tending to infinity such that $r_1 < |f(x_1)| < r_2 < |f(x_2)| < \cdots < r_i < |f(x_i)| < \cdots$. Since X has a basis of open-and-closed sets and f is continuous, there exist open-and-closed sets V_i with $x_i \in V_i \subseteq \{x \in X \mid r_i < |f(x)| < r_{i+1}\}$. In particular, $V_i \cap V_j = \varnothing$ if $i \neq j$. Moreover,

$$V_0 = X - \bigcup_{i=1}^{\infty} V_i = \bigcup_{i=1}^{\infty} (\{x \in X \mid |f(x)| < r_{i+1}\} - (V_1 \cup V_2 \cup \cdots \cup V_i))$$

is open. Thus, X admits an infinite partition, so that it is not a *ZPC* space by 1.8.2.

COROLLARY 1.9.4. *An arbitrary space X is Z-pseudocompact if and only if $\theta(X)$ is pseudocompact.*

2. **Ideals in $C(X, Z)$.** In the theory of $C(X)$, an important part is played by the correspondence between the ideals in $C(X)$ and the filters in the lattice of zero sets of continuous functions. The purpose of this section is to develop an analogous correspondence for $C(X, Z)$. Throughout this section, X is assumed to be a fixed topological space.

2.1. Let Π be the set of all rational primes. We will consider Π to be a T_1-topological space with the closed sets Φ being precisely the finite (or empty) subsets, or all of Π. This is the standard topology for Π considered as the structure space of the ring Z (see [4, p. 204]). We will be concerned with the product space (or set) $X \times \Pi$. It is convenient to introduce standard symbols ξ and π for the projection mappings of $X \times \Pi$ onto X and Π respectively: $\xi\langle x, p \rangle = x$, $\pi\langle x, p \rangle = p$.

DEFINITION 2.1.1. Let $f \in C(X, Z)$. The *divisibility set* of f, denoted $\boldsymbol{D}(f)$, is defined to be $\{\langle x, p \rangle \in X \times \Pi \mid f(x) \equiv 0 \bmod p\}$. The collection of all divisibility sets of elements of $C(X, Z)$ is denoted by $\mathfrak{D}$ (or $\mathfrak{D}(X)$ if it is necessary to indicate the dependence on X).

LEMMA 2.1.2. $\mathfrak{D}$ *is the collection of all sets of the form $\bigcup_{i=1}^{\infty} V_i \times \Phi_i$, where $\{V_i\}$ is a partition of X and each Φ_i is a closed subset of Π.*

Proof. Let $f \in C(X, Z)$. Define $U_n = \{x \in X \mid f(x) = n\}$, $\Psi_n = \{p \in \Pi \mid p$ divides $n\}$ (hence $\Psi_1 = \varnothing$ and $\Psi_0 = \Pi$). Clearly $\boldsymbol{D}(f) = \bigcup_{n \in Z} U_n \times \Psi_n$, which is

the desired form with a suitable rearrangement of subscripts. Conversely, suppose that $D = \bigcup_{i=1}^{\infty} V_i \times \Phi_i$. Define $f_i(x) = 0$ if $x \in V_i$ and $\Phi_i = \Pi$, $f_i(x) = 1$ if $x \in V_i$ and $\Phi_i = \varnothing$, and $f_i(x) = p_1 p_2 \cdots p_j$ if $x \in V_i$ and $\Phi_i = \{p_1, p_2, \cdots, p_j\}$. Then $f \in C(X, Z)$ by 1.2 and by the definition of f, $D(f) = D$.

As usual, the mapping D induces a mapping from the subsets of $C(X, Z)$ to the subsets of $\mathfrak{D}$, namely $D(F) = \{D(f) \mid f \in F\}$, and from the subsets of $\mathfrak{D}$ to the subsets of $C(X, Z)$, that is $D^{-1}(\mathfrak{F}) = \{f \in C(X, Z) \mid D(f) \in \mathfrak{F}\}$.

DEFINITION 2.1.3. Define $\mathfrak{B}$ to be the subset of $\mathfrak{D}$ consisting of all $D(f)$ such that $f(x) \neq 0$ for all $x \in X$.

LEMMA 2.1.4. $\mathfrak{B}$ *is the collection of all sets of the form* $\bigcup_{i=1}^{\infty} V_i \times \Phi_i$, *where* $\{V_i\}$ *is a partition of X and each Φ_i is a finite subset of Π.*

The proof is like that of 2.1.2.

COROLLARY 2.1.5. *If $B \in \mathfrak{B}$ and if A is a subset of $X \times \Pi$ which is of the form* $\bigcup_{i=1}^{\infty} V_i \times \Pi_i$, *where $\{V_i\}$ is a partition of X and each Π_i is an arbitrary subset of Π, then $B \cap A \in \mathfrak{B}$.*

COROLLARY 2.1.6. *If $B \in \mathfrak{B}$ and $D \in \mathfrak{D}$, then $B \cap D^c \in \mathfrak{B}$ (the complement being formed with respect to $X \times \Pi$).*

It is convenient to introduce notation for the cross-sections of subsets of $X \times \Pi$. For any $A \subseteq X \times \Pi$ and any $x \in X$, denote

$$A_x = \{p \in \Pi \mid \langle x, p \rangle \in A\}.$$

Note that if $D \in \mathfrak{D}$, then for any x, D_x is either finite or all of Π. Moreover, D_x is finite for all x if and only if $D \in \mathfrak{B}$. This implies:

LEMMA 2.1.7. *Suppose that $C \subseteq X \times \Pi$, $D \in \mathfrak{D}$, $C \cap D = \varnothing$ and $C_x \neq \varnothing$ for all $x \in X$. Then $D \in \mathfrak{B}$.*

2.2. The lemma of this article is a simple consequence of the definition of D and 1.3.2.

LEMMA 2.2.1. *Let f, $g \in C(X, Z)$. Then*
(a) $D(fg) = D(f) \cup D(g)$;
(b) $D(f - g) \supseteq D(f) \cap D(g)$;
(c) $D((f, g)) = D(f) \cap D(g)$.

COROLLARY 2.2.2. $\mathfrak{D}$ *is closed under set unions and intersections. $\mathfrak{B}$ is a Boolean ring of sets.*

2.3. We now use D and D^{-1} to establish a correspondence between the ideals of $C(X, Z)$ and the filters of $\mathfrak{D}$.

LEMMA 2.3.1. *If J is an ideal of $C(X, Z)$, then $D(J)$ is a filter of $\mathfrak{D}$. If $\mathfrak{J}$ is a filter of $\mathfrak{D}$, then $D^{-1}(\mathfrak{J})$ is an ideal of $C(X, Z)$.*

This is clear from 2.2.1. The next two lemmas are also obvious.

LEMMA 2.3.2. $J_1 \subseteq J_2$ implies $D(J_1) \subseteq D(J_2)$; $\mathcal{J}_1 \subseteq \mathcal{J}_2$ implies $D^{-1}(\mathcal{J}_1) \subseteq D^{-1}(\mathcal{J}_2)$; $D^{-1}(D(J)) \supseteq J$; $D(D^{-1}(\mathcal{J})) = \mathcal{J}$.

LEMMA 2.3.3. If $\{\mathcal{J}_\sigma \mid \sigma \in S\}$ is a set of filters of $\mathfrak{D}$, then

$$D^{-1}\left(\bigcap_{\sigma \in S} \mathcal{J}_\sigma\right) = \bigcap_{\sigma \in S} D^{-1}(\mathcal{J}_\sigma).$$

LEMMA 2.3.4. If J is a proper ideal of $C(X, Z)$, then $D(J)$ is a proper filter. If $\mathcal{J}$ is a proper filter, then $D^{-1}(\mathcal{J})$ is a proper ideal of $C(X, Z)$.

Proof. If $D(f) = \varnothing$, then $f(x) = \pm 1$ for all $x \in X$. Therefore $f^2 = 1$ and if J is proper, then $f \notin J$. Obviously, if $\varnothing \notin \mathcal{J}$, then $1 \notin D^{-1}(\mathcal{J})$.

COROLLARY 2.3.5. If M is a maximal (proper) ideal of $C(X, Z)$, then $D(M)$ is an ultrafilter of $\mathfrak{D}$. If $\mathfrak{M}$ is an ultrafilter of $\mathfrak{D}$, then $D^{-1}(\mathfrak{M})$ is a maximal ideal of $C(X, Z)$.

Proof. Suppose that $D(M) \subseteq \mathcal{J} \subset \mathfrak{D}$. Then $M \subseteq D^{-1}(D(M)) \subseteq D^{-1}(\mathcal{J}) \subset C(X, Z)$. Since M is maximal, $M = D^{-1}(\mathcal{J})$. Therefore $\mathcal{J} = D(D^{-1}(\mathcal{J})) = D(M)$. Suppose that $D^{-1}(\mathfrak{M}) \subseteq J \subset C(X, Z)$. Then $\mathfrak{M} = D(D^{-1}(\mathfrak{M})) \subseteq D(J) \subset \mathfrak{D}$. Since $\mathfrak{M}$ is an ultrafilter, $\mathfrak{M} = D(J)$. Therefore, $J \subseteq D^{-1}(D(J)) = D^{-1}(\mathfrak{M}) \subseteq J$.

COROLLARY 2.3.6. If M is a maximal ideal, then $D^{-1}(D(M)) = M$.

2.4. We wish now to prove that every proper filter in $\mathfrak{D}$ is an intersection of ultrafilters. The proof of this fact is based on a lemma which is useful in its own right. For any $A \in \mathfrak{D}$, define $\mathfrak{D}_A = \{D \cap A^c \mid D \in \mathfrak{D}\}$. More generally, if $\mathfrak{F} \subseteq \mathfrak{D}$, let $\mathfrak{F}_A = \{D \cap A^c \mid D \in \mathfrak{F}\}$. Evidently $\mathfrak{D}_A$ is closed under intersections and $\varnothing \in \mathfrak{D}_A$. Thus, it makes sense to talk about filters in $\mathfrak{D}_A$.

LEMMA 2.4.1. Let $A \in \mathfrak{D}$ and $A \neq X \times \Pi$. Suppose that $\mathfrak{N}$ is an ultrafilter of $\mathfrak{D}_A$. Then $\mathfrak{N} \cap \mathfrak{B} \neq \varnothing$.

Proof. Let $A = D(f)$, $f \in C(X, Z)$. Define $V_n = \{x \in X \mid f(x) = n\}$. Define $g \in C(X, Z)$ by $g(x) = 1$ for $x \in V_0$, and for $x \in V_n$ ($n \neq 0$), let $g(x) = p_n$, where p_n is any prime not dividing n. Then, by definition of g, $D(g) \in \mathfrak{B}$ and $D(g) \subseteq A^c$. If $D(g) \in \mathfrak{N}$ we are through. Otherwise, since $\mathfrak{N}$ is an ultrafilter, there exists $D \in \mathfrak{D}$ such that $D \cap A^c \in \mathfrak{N}$ and $D(g) \cap D \cap A^c = \varnothing$. Since $D(g) \subseteq A^c$, this implies $D(g) \cap D = \varnothing$. In particular, $(D(g))_x \cap D_x = \varnothing$ for all x. Therefore $D_x \neq \Pi$, unless $(D(g))_x = \varnothing$, that is, $g(x) = 1$. But if $g(x) = 1$, then $x \in V_0$ and $A_x = \Pi$. Thus, for all $x \in X$, $(D \cap A^c)_x$ is finite. Consequently, $D \cap A^c \in \mathfrak{B}$.

LEMMA 2.4.2. Every proper filter in $\mathfrak{D}$ is the intersection of ultrafilters.

Proof. Suppose that $\mathcal{J}$ is a filter of $\mathfrak{D}$ and A is an element of $\mathfrak{D}$ which does not belong to $\mathcal{J}$. It will suffice to show that there is an ultrafilter $\mathfrak{M}$ of $\mathfrak{D}$ such that $\mathcal{J} \subseteq \mathfrak{M}$ and $A \notin \mathfrak{M}$. It is easy to see that $\mathcal{J}_A$ is a filter in $\mathfrak{D}_A$. Since $A \notin \mathcal{J}$, $\mathcal{J}_A$ is proper. Let $\mathfrak{N}$ be an ultrafilter of $\mathfrak{D}_A$ with $\mathcal{J}_A \subseteq \mathfrak{N}$. Define $\mathfrak{M} = \{D \in \mathfrak{D} \mid D \cap A^c \in \mathfrak{N}\}$. Clearly $\mathfrak{M}$ is a filter of $\mathfrak{D}$ which does not contain A. Suppose that $C \in \mathfrak{D} - \mathfrak{M}$. Then $C \cap A^c \notin \mathfrak{N}$. Since $\mathfrak{N}$ is an ultrafilter of $\mathfrak{D}_A$, there exists $D \in \mathfrak{M}$ such that $(C \cap A^c) \cap (D \cap A^c) = \varnothing$. By 2.4.1, there exists $B \in \mathfrak{N} \cap \mathfrak{B}$. Then $B \cap D \in \mathfrak{M}$ and $C \cap (B \cap D) = C \cap D \cap A^c \cap B = \varnothing$. Since C was arbitrary, $\mathfrak{M}$ is an ultrafilter.

2.5. We can now easily prove the main theorem of this section.

THEOREM 2.5.1. *If J is an ideal of $C(X, Z)$, then $D^{-1}(D(J))$ is the intersection of all maximal ideals of $C(X, Z)$ which contain J.*

Proof. Let M be a maximal ideal containing J. Then by 2.3.2 and 2.3.6, $M = D^{-1}(D(M)) \supseteq D^{-1}(D(J))$. Thus, $D^{-1}(D(J))$ is contained in the intersection of all maximal ideals containing J. On the other hand, by 2.3.4 and 2.4.2, we can write $D(J) = \bigcap_{\sigma \in S} \mathfrak{M}_\sigma$, where $\{\mathfrak{M}_\sigma \mid \sigma \in S\}$ is a set of ultrafilters of $\mathfrak{D}$. Therefore, by 2.3.2, 2.3.3 and 2.3.5, $D^{-1}(D(J)) = \bigcap_{\sigma \in S} D^{-1}(\mathfrak{M}_\sigma)$ is an intersection of maximal ideals containing J.

COROLLARY 2.5.2. *The following conditions are equivalent for an ideal J of $C(X, Z)$:*
    (a) $D^{-1}(D(J)) = J$;
    (b) J is an intersection of maximal ideals;
    (c) $C(X, Z)/J$ is a subdirect sum of fields;
    (d) $C(X, Z)/J$ has zero Jacobson radical.
    See [4, p. 14].

Defining the circle composition as usual by $f \circ g = f + g - fg$, we easily deduce from [4, p. 9]:

COROLLARY 2.5.3. *If J is an ideal of $C(X, Z)$, then $D^{-1}(D(J)) = \{f \in C(X, Z) \mid$ for all g, there exists h such that $h \circ (gf) \in J\}$.*

2.6. By 2.5.1, $D^{-1}(D(J))$ is the Jacobson radical of the ideal J. It can be inferred that $D^{-1}(D(J))$ contains the ordinary radical of J: $J^{1/2} = \{f \in C(X, Z) \mid f^k \in J$ for some $k = 1, 2, \cdots \}$.

THEOREM 2.6.1. *If J is an ideal of $C(X, Z)$, then $D^{-1}(D(J)) \supseteq J^{1/2}$. The equality holds for all ideals J of $C(X, Z)$ if and only if X is a ZPC space.*

Proof. If $f^k \in J$, then $D(f) = D(f^k) \in D(J)$ and therefore $f \in D^{-1}(D(J))$. This shows, in a very direct way, that $J^{1/2} \subseteq D^{-1}(D(J))$. Suppose that X is a ZPC space. Then if $f \in D^{-1}(D(J))$, there is a function $g \in J$ such that $D(f) = D(g)$. That is, for each $x \in X$, the prime divisors of $f(x)$ coincide with those of $g(x)$. Hence, if k is an integer greater than all exponents of the prime

divisors of $g(x)$, then $f(x)^k$ is divisible by $g(x)$. Since X is a ZPC space, g takes only finitely many values and it is possible to choose k so large that $g(x)$ divides $f(x)^k$ for all $x \in X$. Thus, by 1.3.1, $f^k = gh \in J$ and consequently $f \in J^{1/2}$. Finally, suppose that X is not a ZPC space. Let $\{V_j\}$ be an infinite partition of X. Let p be any prime. Define $f \in C(X, Z)$ by $f(x) = p^j$ for all $x \in V_j$. Let $J = \{fh \mid h \in C(X, Z)\}$. By definition of f, $D(f) = D(p)$. Thus, $p \in D^{-1}(D(J))$. However, for each k, p^k is not divisible by f in $C(X, Z)$. Thus, $p \notin J^{1/2}$.

3. **The structure space of** $C(X, Z)$. Throughout this section X is a fixed topological space.

3.1. The structure space of $C(X, Z)$ is defined as for any commutative ring with identity to be the set $\mathfrak{S} = \mathfrak{S}(X)$ of all maximal ideals topologized by the hull-kernel closure operation: for $\mathfrak{A} \subseteq \mathfrak{S}$

$$\mathfrak{A}^- = \{M \in \mathfrak{S} \mid M \supseteq \cap \mathfrak{A}\}.$$

This closure operation makes $\mathfrak{S}$ into a compact T_1 space (see [4, pp. 204 and 208]). The importance of the structure space $\mathfrak{S}$ resides in an elementary consequence of 2.5.1.

LEMMA 3.1.1. *There is a one-to-one correspondence* (*order reversing*) *between the closed subsets of* $\mathfrak{S}$ *and the ideals* J *of* $C(X, Z)$ *satisfying* $D^{-1}(D(J)) = J$. *This mapping is defined by* $\mathfrak{A} \to \cap \mathfrak{A}$.

It is possible to introduce a similar topology on the set $\mathfrak{S}_1$ of all ultra-filters of $\mathfrak{D}$: if $\mathfrak{A}_1 \subseteq \mathfrak{S}_1$, then

$$\mathfrak{A}_1^- = \{\mathfrak{M} \in \mathfrak{S}_1 \mid \mathfrak{M} \supseteq \cap \mathfrak{A}_1\}.$$

It is easily shown that this closure operation makes $\mathfrak{S}_1$ into a compact T_1 space also.

LEMMA 3.1.2. *The mapping* D *is a homeomorphism of* $\mathfrak{S}$ *onto* $\mathfrak{S}_1$.

Proof. By 2.3.2, 2.3.5 and 2.3.6, D maps $\mathfrak{S}$ one-to-one onto $\mathfrak{S}_1$, with inverse D^{-1}. It suffices to show that D^{-1} commutes with the closure operations. Let $\mathfrak{A}_1 \subseteq \mathfrak{S}_1$. Then $\mathfrak{M} \in \mathfrak{A}_1^-$ if and only if $\mathfrak{M} \supseteq \cap \mathfrak{A}_1$. By 2.3.3, this is equivalent to $D^{-1}(\mathfrak{M}) \supseteq D^{-1}(\cap \mathfrak{A}_1) = \cap D^{-1} \mathfrak{A}_1$. Hence $D^{-1}(\mathfrak{A}_1^-) = (D^{-1}\mathfrak{A}_1)^-$ and therefore D^{-1} is a homeomorphism. Thus, D is a homeomorphism.

3.2. We wish now to determine the structure space of $C(X, Z)$ in case X is a ZPC space. By 3.1.1, it is sufficient to consider $\mathfrak{S}_1$.

LEMMA 3.2.1. *Suppose* X *is compact and totally disconnected. Then the mapping* $\phi: \langle x, p \rangle \to \{D \in \mathfrak{D} \mid \langle x, p \rangle \in D\}$ *is a homeomorphism of* $X \times \Pi$ *onto* $\mathfrak{S}_1$.

Proof. The proof is based on three easily verified observations. (a) The sets of $\mathfrak{D}$ are closed; (b) $X \times \Pi$ is compact; (c) the sets of $\mathfrak{D}$ separate points. Note that $\{D \in \mathfrak{D} \mid \langle x, p \rangle \in D\} = D(\{f \in C(X, Z) \mid f(x) \equiv 0 \mod p\})$. Since

$\{f \in C(X, Z) | f(x) \equiv 0 \bmod p\}$ is a maximal ideal, namely the kernel of the homomorphism $f \to f(x) + pZ$ of $C(X, Z)$ on Z_p, we see that $\{D \in \mathfrak{D} | \langle x, p \rangle \in D\}$ is an ultrafilter of $\mathfrak{D}$. Suppose $\mathfrak{M} \in \mathfrak{S}_1$. By (a) and (b), $E = \cap \mathfrak{M}$ is not empty. Let $\langle x, p \rangle \in E$. Then $C \in \mathfrak{M}$ implies $\langle x, p \rangle \in C$ and therefore $C \in \{D \in \mathfrak{D} | \langle x, p \rangle \in D\}$. Consequently, since $\mathfrak{M}$ is an ultrafilter, $\mathfrak{M} = \{D \in \mathfrak{D} | \langle x, p \rangle \in D\}$. Thus, ϕ is onto. By (c), ϕ is one-to-one. Suppose that $\mathfrak{A}_1$ is closed in $\mathfrak{S}_1$. Let $\langle x, p \rangle \notin \phi^{-1}(\mathfrak{A}_1)$. That is, $\{D \in \mathfrak{D} | \langle x, p \rangle \in D\}$ does not contain $\cap \mathfrak{A}_1$. Hence, there is a $C \in \cap \mathfrak{A}_1$ such that $\langle x, p \rangle \notin C$. Suppose $C = \cap_i V_i \times \Phi_i$, where $\{V_i\}$ is a partition of X. Then $x \in V_i$ for some i. Since $\langle x, p \rangle \notin C$, this implies $p \notin \Phi_i$. Therefore Φ_i is finite and $V_i \times \Phi_i^c$ is a neighborhood of $\langle x, p \rangle$ in $X \times \Pi$. Moreover, if $\langle y, q \rangle \in V_i \times \Phi_i^c$, then $\langle y, q \rangle \notin C$. Hence $C \notin \phi(\langle y, q \rangle)$ and therefore since $C \in \cap \mathfrak{A}_1$, $\phi(\langle y, q \rangle) \notin \mathfrak{A}_1$. Hence $V_i \times \Phi_i^c \subseteq (\phi^{-1}(\mathfrak{A}_1))^c$. Thus, $\phi^{-1}(\mathfrak{A}_1)$ is closed and ϕ is continuous. Since X is totally disconnected, the sets of the form $V \times \Phi^c$, where V is open-and-closed in X and $\Phi \subseteq \Pi$ is finite, constitute a basis for the open sets of $X \times \Pi$. If $\langle x, p \rangle \notin V \times \Phi^c$, then $\langle x, p \rangle \in V \times \Phi$ or $\langle x, p \rangle \in V^c \times \Pi$ and conversely. Thus, $\phi((V \times \Phi^c)^c) = \{\mathfrak{M} \in \mathfrak{S}_1 | V \times \Phi \in \mathfrak{M}\} \cup \{\mathfrak{M} \in \mathfrak{S}_1 | V^c \times \Pi \in \mathfrak{M}\}$. Since the sets $\{\mathfrak{M} \in \mathfrak{S}_1 | V \times \Phi \in \mathfrak{M}\}$ and $\{\mathfrak{M} \in \mathfrak{S}_1 | V^c \times \Pi \in \mathfrak{M}\}$ are obviously closed in $\mathfrak{S}_1$, it follows that ϕ is bicontinuous.

Combining 3.1.1 and 3.2.1 with 1.5.2 and 1.8.1 yields

COROLLARY 3.2.2. *Let X be a ZPC space. Then $\mathfrak{S}$ is homeomorphic to $\delta X \times \Pi$.*

COROLLARY 3.2.3. *If X is a ZPC space, then every residue class field of $C(X, Z)$ is isomorphic to Z_p for some prime p.*

Proof. Since $C(X, Z) \cong C(\delta X, Z)$, we may suppose that X is compact and totally disconnected. Then, as we noted in the proof of 3.2.1, every maximal ideal of $C(X, Z)$ is the kernel of a homomorphism of $C(X, Z)$ on Z_p.

COROLLARY 3.2.4. *If X is any space, then every residue class field of $C^*(X, Z)$ is isomorphic to Z_p for some prime p.*

Proof. By 1.5.2, $C^*(X, Z) \cong C^*(\delta X, Z) = C(\delta X, Z)$. Thus, 3.2.4 follows from 3.2.3.

3.3. If X is not a ZPC space, then the structure space of $C(X, Z)$ is more complicated. We devote the remainder of this section to the study of this situation. First it is convenient to introduce some notation.

DEFINITION 3.3.1. Let X be any space. Let $\mathfrak{S}$ be the structure space of $C(X, Z)$. Define

$$\mathfrak{P}(= \mathfrak{P}(X)) = \{M \in \mathfrak{S} | C(X, Z)/M \text{ has prime characteristic}\},$$

$$\mathfrak{Z}(= \mathfrak{Z}(X)) = \{M \in \mathfrak{S} | C(X, Z)/M \text{ has zero characteristic}\}.$$

LEMMA 3.3.2. *If P is a proper prime ideal of $C(X, Z)$ such that the character-istic of $C(X, Z)/P$ is a prime p, then $C(X, Z)/P \cong Z_p$.*

Proof. The constant function p belongs to P. If $f \in C(X, Z)$, then the function $(f-1)(f-2) \cdots (f-p)$ is divisible by p at each point. Hence, it is divisible by p because of 1.3.1. Therefore $(f-1)(f-2) \cdots (f-p)$ is in P. Since P is prime, $f-k \in P$ for some k, $1 \leq k \leq p$. That is, the constant functions modulo p form a complete system of residues of $C(X, Z)$ modulo P.

A similar argument shows that if P is any prime ideal of $C(X, Z)$ then each bounded function of $C(X, Z)$ is congruent modulo P to a constant function on X.

LEMMA 3.3.3. *Let M be a maximal ideal of $C(X, Z)$. Then the following conditions are equivalent.*
   (a) $C(X, Z)/M \cong Z_p$;
   (b) $C(X, Z)/M$ *has characteristic* p;
   (c) M *contains* p;
   (d) $D(M)$ *contains* $X \times \{p\}$;
   (e) $D \in D(M)$ *implies* $D \cap (X \times \{p\}) \neq \varnothing$.

Proof. By 3.3.2, (a) is equivalent to (b). Obviously, (b) is equivalent to (c) and (c) is equivalent to (d) by 2.3.6. Finally, (d) is equivalent to (e) by 2.3.5.

COROLLARY 3.3.4. *Let M be a maximal ideal of $C(X, Z)$. Then the following conditions are equivalent.*
   (a) $M \in \mathfrak{P}$;
   (b) M *contains* n *for some nonzero* $n \in Z$;
   (c) M *contains a bounded function* f *such that* $f(x) \neq 0$ *for all* $x \in X$.

Proof. Clearly (a) implies (b) implies (c). If (c) is satisfied, then by 1.2, M contains n, where n is the least common multiple of all the integers $f(x)$, $x \in X$. Finally (b) implies that the characteristic of $C(X, Z)/M$ is not zero.

3.4. LEMMA. *Let M^* be a maximal ideal of $C^*(X, Z)$. Define $M = M^*C(X, Z)$. Then M is a maximal ideal of $C(X, Z)$ such that $M^* = M \cap C^*(X, Z)$.*

Proof. First we show that M is a proper ideal of $C(X, Z)$. By definition, $M^*C(X, Z) = \{\Sigma_i f_i g_i \mid f_i \in M^*, \ g_i \in C(X, Z)\}$, which is clearly an ideal. If $\Sigma_i f_i g_i \in M^*C(X, Z)$, then $D(\Sigma_i f_i g_i) \supseteq \cap_i D(f_i) \neq \varnothing$. Thus, by 2.3.4 and 2.3.2, M is proper. Consequently, $M \cap C^*(X, Z)$ is a proper ideal of $C^*(X, Z)$ containing the maximal ideal M^*. Therefore $M \cap C^*(X, Z) = M^*$. By 3.2.4, $C^*(X, Z)/M^* \cong Z_p$ for some prime p. Consequently M contains p. Hence by 1.3.1, $C(X, Z) = M + C^*(X, Z)$. Thus, $C(X, Z)/M = (C^*(X, Z)+M)/M \cong C^*(X, Z)/(M \cap C^*(X, Z)) = C^*(X, Z)/M^* \cong Z_p$. Thus M is a maximal ideal.

3.5. We can now relate the subspace $\mathfrak{P}$ of $\mathfrak{S}$ to the structure space of $C^*(X, Z)$.

THEOREM 3.5.1. *For $M \in \mathfrak{P}$, define $\phi(M) = M \cap C^*(X, Z)$. Then ϕ maps $\mathfrak{P}$ one-to-one and continuously onto the structure space of $C^*(X, Z)$.*

Proof. If $M \in \mathfrak{P}$, then for some p, $C(X, Z)/M \cong Z_p$ by 3.3.3. Restricted to $C^*(X, Z)$, the mapping of $C(X, Z)$ on Z_p with kernel M will have kernel $M \cap C^*(X, Z)$. Thus, $\phi(M)$ is a maximal ideal of $C^*(X, Z)$. By 3.4, $(M \cap C^*(X, Z))C(X, Z)$ is a maximal ideal of $C(X, Z)$ and it is clearly contained in M. Thus, $(M \cap C^*(X, Z))C(X, Z) = M$. Together with 3.4, this shows that ϕ is a one-to-one mapping of $\mathfrak{P}$ onto the structure space of $C^*(X, Z)$. Let $\mathfrak{A}^*$ be a closed subset of the structure space of $C^*(X, Z)$. We wish to show that $\mathfrak{A} = \{ M \in \mathfrak{P} \,|\, M \cap C^*(X, Z) \in \mathfrak{A}^* \}$ is closed in $\mathfrak{P}$, that is, $\mathfrak{A}^- \cap \mathfrak{P} \subseteq \mathfrak{A}$. Let $M \in \mathfrak{A}^- \cap \mathfrak{P}$. Then $M \supseteq \cap \mathfrak{A}$. Since $N \in \mathfrak{A}$ implies $N \cap C^*(X, Z) \in \mathfrak{A}^*$, it follows that $M \cap C^*(X, Z) \supseteq \cap \mathfrak{A}^*$. Hence, since $\mathfrak{A}^*$ is closed, $M \cap C^*(X, Z) \in \mathfrak{A}^*$ and therefore $M \in \mathfrak{A}$.

If X is not a ZPC space, then the mapping ϕ of 3.5.1 is not bicontinuous. In fact if f is an unbounded function of $C(X, Z)$, then the set $\mathfrak{A} = \{ M \in \mathfrak{P} \,|\, f \in M \}$ is clearly closed in $\mathfrak{P}$, but $\phi(\mathfrak{A})$ is not closed in the structure space of $C^*(X, Z)$.

COROLLARY 3.5.2. *For any space X, there is a continuous, one-to-one mapping of $\mathfrak{P}(X)$ onto $\delta X \times \Pi$.*

4. Existence theorems for maximal ideals. The purpose of this section is to establish the existence of maximal ideals in $C(X, Z)$ having various special properties.

4.1. We will associate with each maximal ideal M of $C(X, Z)$ an ultrafilter $\mathfrak{u}(M)$ in $\mathfrak{B}(X)$ and an ultrafilter $\mathcal{P}(M)$ in the Boolean algebra of all subsets of Π. It will be shown in §6 that $\mathcal{P}(M)$ is closely related to the algebraic properties of the residue class field $C(X, Z)/M$.

DEFINITION 4.1.1. Let M be a maximal ideal of $C(X, Z)$. Define

$$\mathfrak{u}(M) = \xi(D(M)).$$

$$\mathcal{P}(M) = \{ \Pi_0 \subseteq \Pi \,|\, \Pi_0 \supseteq \pi(D(f)) \text{ for some } f \in M \}.$$

LEMMA 4.1.2. *$\mathfrak{u}(M)$ is an ultrafilter in $\mathfrak{B}(X)$.*

Proof. If $f \in C(X, Z)$, then $\xi(D(f)) = \{ x \in X \,|\, f(x) \neq 1 \}$. Thus, $\xi(D(M)) \subseteq \mathfrak{B}(X)$. We show that for $V \in \mathfrak{B}(X)$ to be in $\mathfrak{u}(M)$ it is necessary and sufficient that $V \times \Pi \in D(M)$. The sufficiency is clear. If $V \times \Pi \notin D(M)$, then $V^c \times \Pi \in D(M)$. Hence, no $D \in \mathfrak{D}$ such that $\xi(D) = V$ can be in $D(M)$ since $D \cap (V^c \times \Pi) = \varnothing$. This shows that $V \notin \mathfrak{u}(M)$. It is now evident that $\mathfrak{u}(M)$ is an ultrafilter.

If X is a ZPC space, then $\mathcal{P}(M)$ is the fixed ideal of all subsets of Π containing a prime p by 3.2.3 and 3.3.3. Hence, in this case, $\mathcal{P}(M)$ is an ultrafilter in the Boolean algebra of all subsets of Π. To prove that $\mathcal{P}(M)$ is an ultrafilter if X is not a ZPC space, we need a preliminary result.

LEMMA 4.1.3. *If X is not a ZPC space and Π_0 is any subset of Π, then there exists $f \in C(X, Z)$ such that $\pi(D(f)) = \Pi_0$ and either $f = 1$ or $f(x)$ is prime for all $x \in X$.*

Proof. If Π_0 is empty, let $f = 1$. Hence suppose that $\Pi_0 = \{p_1, p_2, \cdots\}$, with repetitions allowed if Π_0 is finite. Let $\{V_i\}$ be an infinite partition of X with $V_i \neq \varnothing$ for all i. Define $f(x) = p_i$ for $x \in V_i$. Clearly, $\pi(D(f)) = \Pi_0$.

THEOREM 4.1.4. $\mathcal{P}(M)$ *is an ultrafilter in the Boolean algebra of all subsets of* Π. *If X is not a ZPC space, then* $\mathcal{P}(M) = \pi(D(M))$.

Proof. We can assume that X is not a ZPC space. Suppose $\Pi_1 \in \pi(D(M))$ and $\Pi_2 \supseteq \Pi_1$. Then $\Pi_1 = \pi(D(f))$ for some $f \in M$. Let $g \in C(X, Z)$ be such that $\pi(D(g)) = \Pi_2$. Then $fg \in M$ and $\pi(D(fg)) = \pi(D(f) \cup D(g)) = \Pi_1 \cup \Pi_2 = \Pi_2$. Thus, $\Pi_2 \in \pi(D(M))$. This shows that $\mathcal{P}(M) = \pi(D(M))$. Now suppose that $\Pi_1, \Pi_2 \in \mathcal{P}(M)$, say $\Pi_1 = \pi(D(f))$, $\Pi_2 = \pi(D(g))$, with $f, g \in M$. Then $\Pi_1 \cap \Pi_2 \supseteq \pi(D(f) \cap D(g)) = \pi(D((f, g)))$ and $(f, g) \in M$. Therefore $\Pi_1 \cap \Pi_2 \in \mathcal{P}(M)$. Finally, if Π_0 is any subset of Π, then either $\Pi_0 \in \mathcal{P}(M)$ or $\Pi_0^c \in \mathcal{P}(M)$. To see this, let $f \in M$ be such that $D(f) \in \mathcal{B}$. The existence of such an f follows from the case $A = \varnothing$ of 2.4.1 (with $\mathfrak{N} = D(M)$). By 4.1.3, there exists $g \in C(X, Z)$ such that $D(g) \in \mathcal{B}$ and $\pi(D(g)) = \Pi$. Then $fg \in M$, $D(fg) \in \mathcal{B}$ and $\pi(D(fg)) = \Pi$. By 2.1.5, $D(fg) \cap (X \times \Pi_0) \in \mathcal{B}$ and $D(fg) \cap (X \times \Pi_0^c) \in \mathcal{B}$. Since $(D(fg) \cap (X \times \Pi_0)) \cup (D(fg) \cap (X \times \Pi_0^c)) = D(fg) \in D(M)$, it follows that either $D(fg) \cap (X \times \Pi_0) \in D(M)$, or $D(fg) \cap (X \times \Pi_0^c) \in D(M)$. Thus, since $\pi(D(fg)) = \Pi$, either $\Pi_0 = \pi(D(fg) \cap (X \times \Pi_0)) \in \pi(D(M)) = \mathcal{P}(M)$, or $\Pi_0^c = \pi(D(fg) \cap (X \times \Pi_0^c)) \in \pi(D(M)) = \mathcal{P}(M)$.

4.2. We wish to show that every ultrafilter $\mathcal{P}$ in the Boolean algebra of all subsets of Π and every ultrafilter $\mathfrak{U}$ in $\mathfrak{B}(X)$ can be realized as $\mathcal{P}(M)$ and $\mathfrak{U}(M)$ respectively. In order to do this, we introduce two methods of constructing maximal ideals in $C(X, Z)$.

DEFINITION 4.2.1. Let $\mathcal{P}$ be an ultrafilter in the Boolean algebra of all subsets of Π. Let $f \in C(X, Z)$ be such that $f(x) \neq 0$ for all x and $\pi(D(f)) \in \mathcal{P}$. Define

$$J(f, \mathcal{P}) = \{g \in C(X, Z) \mid D(g) \supseteq D(f) \cap (X \times \Pi_0), \text{ some } \Pi_0 \in \mathcal{P}\}.$$

LEMMA 4.2.2. $J(f, \mathcal{P})$ *is a proper ideal of $C(X, Z)$. If M is a maximal ideal containing $J(f, \mathcal{P})$, then $\mathcal{P}(M) = \mathcal{P}$.*

Proof. Since $f(x) \neq 0$ for all x, $D(f) \cap (X \times \Pi_0) \in \mathfrak{D}$ by 2.1.5. By definition, $J(f, \mathcal{P}) = D^{-1}(\mathfrak{g})$, where $\mathfrak{g}$ is the filter in $\mathfrak{D}$ which is generated by $\{D(f) \cap (X \times \Pi_0) \mid \Pi_0 \in \mathcal{P}\}$. Notice that $\pi(D(f) \cap (X \times \Pi_0)) = \pi(D(f)) \cap \Pi_0 \neq \varnothing$, since $\pi(D(f)) \in \mathcal{P}$. It follows from 2.3.1 and 2.3.4 that $J(f, \mathcal{P})$ is a proper ideal of $C(X, Z)$. Moreover, if M is a maximal ideal containing $J(f, \mathcal{P})$, then $\mathcal{P}(M) \supseteq \pi(D(J(f, \mathcal{P}))) \supseteq \{\pi(D(f)) \cap \Pi_0 \mid \Pi_0 \in \mathcal{P}\}$. Thus, $\mathcal{P}(M) = \mathcal{P}$.

By 4.2.2 we infer that $\mathfrak{Z}(X)$ is not empty (see 3.3.1).

COROLLARY 4.2.3. *If X is not a ZPC space, then $\left|\mathcal{B}(X)\right| \geqq 2^{\mathfrak{c}}$. Moreover, if Π_0 is an infinite subset of Π, there exists $M \in \mathcal{B}(X)$ such that $\Pi_0 \in \mathcal{P}(M)$.*

Proof. By 3.3.3, $M \in \mathcal{B}(X)$ if and only if $\mathcal{P}(M)$ is a free ultrafilter in the Boolean algebra of all subsets of Π. But there are precisely $2^{\mathfrak{c}}$ such free ultra-filters (see [2, p. 131]) and any infinite Π_0 is contained in at least one of them.

4.3. The second method of constructing maximal ideals is more direct: from an ultrafilter in $\mathcal{B}(X)$, one obtains a maximal ideal explicitly. This construction is well known, but has apparently never been published.

DEFINITION 4.3.1. A function $f \in C(X, Z)$ is called prime-valued if for all x, $\left|f(x)\right|$ is either one or a prime.

Evidently, f is prime valued if and only if the sets $D(f)_x$ contain at most one prime.

DEFINITION 4.3.2. Let $\mathfrak{u}$ be an ultrafilter in $\mathcal{B}(X)$. Let $f \in C(X, Z)$ be prime-valued and satisfy $\xi(D(f)) \in \mathfrak{u}$. Define

$$M(f, \mathfrak{u}) = \left\{ g \in C(X, Z) \mid D(g) \supseteq D(f) \cap (V \times \Pi), \text{ for some } V \in \mathfrak{u} \right\}.$$

LEMMA 4.3.3. *$M(f, \mathfrak{u})$ is a maximal ideal in $C(X, Z)$. Moreover, $\mathfrak{u}(M(f, \mathfrak{u})) = \mathfrak{u}$.*

Proof. As in the proof of 4.3.2, $M(f, \mathfrak{u})$ is a proper ideal of $C(X, Z)$ and $\xi(D(M(f, \mathfrak{u}))) \supseteq \left\{ \xi(D(f)) \cap V \mid V \in \mathfrak{u} \right\}$. Suppose that $g \in C(X, Z) - M(f, \mathfrak{u})$. Let $V = \xi(D(f) \cap D(g))$. Then since f is prime-valued, $D(f) \cap D(g) = D(f) \cap (V \times \Pi)$. Hence $V \notin \mathfrak{u}$. Therefore $V^c \in \mathfrak{u}$. Let $h \in C(X, Z)$ be defined by $h(x) = 1$ for $x \in V$ and $h(x) = f(x)$ for $x \in V^c$. Then $D(h) = D(f) \cap (V^c \times \Pi)$, so that $h \in M(f, \mathfrak{u})$. Moreover, $D(g) \cap D(h) = D(f) \cap (V \times \Pi) \cap (V^c \times \Pi) = \varnothing$. Hence, $(g, h) = 1$. Since g was arbitrary, $M(f, \mathfrak{u})$ is maximal. Since $\mathfrak{u}(M(f, \mathfrak{u}))$ is a proper filter containing $\mathfrak{u}$, the last statement of the lemma is clear.

COROLLARY 4.3.4. *If M is a maximal ideal which contains a prime-valued function f, then $M = M(f, \mathfrak{u}(M))$.*

Proof. Note that since $f \in M$, $\xi(D(f)) \in \mathfrak{u}(M)$. Suppose that $h \in M(f, \mathfrak{u}(M))$. Then by definition, there exists $g \in M$ such that $D(h) \supseteq D(f) \cap (\xi(D(g)) \times \Pi) \supseteq D(f) \cap D(g) = D((f, g))$. Thus, $h \in D^{-1}(D(M)) = M$. This shows that $M(f, \mathfrak{u}(M)) \subseteq M$. Since $M(f, \mathfrak{u}(M))$ is maximal by 4.3.3, equality must hold.

4.4. The construction of 4.3.2 provides a method of obtaining all maximal ideals in $C(X, Z)$ which contain a prime-valued function f. However, this does not yield all maximal ideals, as we will now show.

LEMMA. *If X is not a ZPC space, then $C(X, Z)$ contains maximal ideals M such that no prime-valued function in $C(X, Z)$ belongs to M.*

Proof. Let $\left\{ V_i \right\}$ be an infinite partition of X with $V_i \neq \varnothing$ for all i. For each i, let $\Phi_i \subset \Pi$ satisfy $\left|\Phi_i\right| = i$. Define $B = \bigcup_{i=1}^{\infty} V_i \times \Phi_i$. Note that $B \in \mathcal{B}$.

Moreover, if $f_1, f_2, \cdots, f_k$ are prime-valued, then $B \cap (D(f_1))^c \cap (D(f_2))^c$ $\cap \cdots \cap (D(f_k))^c \neq \varnothing$. Indeed, if $i > k$, and if $x \in V_i$, then among the i primes of Φ_i, there must be one which is distinct from the prime divisors of $f_1(x)$, $f_2(x), \cdots, f_k(x)$ (which are at most k in number). If p is such a prime, then $\langle x, p \rangle \in B \cap (D(f_1))^c \cap (D(f_2))^c \cap \cdots \cap (D(f_k))^c$. Let $\mathfrak{M}$ be an ultrafilter of $\mathfrak{D}$ which contains all the sets $B \cap (D(f_1))^c \cap (D(f_2))^c \cap \cdots \cap (D(f_k))^c$, where $f_1, f_2, \cdots, f_k$ range over all prime-valued functions and $k = 1, 2, \cdots$. Note that these sets belong to $\mathfrak{D}$ by 2.1.6. Finally, let $M = D^{-1}(\mathfrak{M})$. Then M is a maximal ideal of $C(X, Z)$. If f is any prime-valued function, then $f \notin M$, since $D(f) \cap (B \cap (D(f))^c) = \varnothing$.

Note that any ideal M which contains no prime-valued function is necessarily an element of $\mathfrak{Z}(X)$ (by 3.3.4).

4.5. In some instances, the existence of a prime-valued function in M simplifies the study of the residue class field $C(X, Z)/M$. In this article, we will show that this simplification can always be achieved at the expense of changing the space X.

LEMMA 4.5.1. *Let $\{V_i\}$ be a partition of X. Then*

$$\phi : C(X, Z) \to \coprod_i C(V_i, Z),$$

defined by $\phi(f) = (\cdots f | V_i \cdots)$, is an isomorphism of $C(X, Z)$ onto the direct product of the rings $C(V_i, Z)$.

LEMMA 4.5.2. *Let $n \neq 0$ and denote by ϕ_n the natural homomorphism of Z onto Z_n. Then the mapping $f \to \phi_n \circ f$ is a homomorphism of $C(X, Z)$ onto $C(X, Z_n)$ with the kernel $nC(X, Z)$. Thus, $C(X, Z)/nC(X, Z) \cong C(X, Z_n)$.*

LEMMA 4.5.3. *Let $p_1, \cdots, p_k$ be distinct primes. Then the mapping*

$$f \to (\phi_{p_1} \circ f, \cdots, \phi_{p_k} \circ f)$$

is a homomorphism of $C(X, Z)$ onto $\prod_{i=1}^k C(X, Z_{p_i})$ with kernel $nC(X, Z)$, where $n = p_1 \cdots p_k$. Hence,

$$C(X, Z_n) \cong \prod_{i=1}^k C(X, Z_{p_i}).$$

These lemmas are easily obtained from 1.2 and 1.3.1.

THEOREM 4.5.4. *Let $F = C(X, Z)/M$, where M is a maximal ideal. Then for a suitable space X_0, there is a maximal ideal M_0 in $C(X_0, Z)$ such that $C(X_0, Z)/M_0 \cong F$ and M_0 contains a prime-valued function.*

Proof. By 2.4.1, M contains a function f such that for all $x \in X$, either $f(x) = 1$, or $f(x)$ is a product of distinct primes. Let $n_1, n_2, \cdots$ be an enumeration of all integers such that the set $V_i = \{x \in X | f(x) = n_i\}$ is not empty. Then each homomorphism of the following sequence is onto:

$$C(X, Z) \to \coprod_i C(V_i, Z) \to \coprod_i (C(V_i, Z)/n_i C(V_i, Z)) \to \coprod_i C(V_i, Z_{n_i}).$$

The mappings of this sequence are the ones defined in 4.5.1 and 4.5.2. It is easy to see that the kernel of the composition of these homomorphisms is precisely $fC(X, Z)$. Thus, by 4.5.3, F is a homomorphic image of $\prod_i C(V_i, Z_{n_i})$ $\cong \prod_i \prod_{j-1}^{k(i)} C(V_i, Z_{p_{ij}})$, where $n_i = p_{i1} \cdots p_{ik(i)}$. Let X_0 be the topological sum of the spaces W_i, which are themselves topological sums $V_{i1} \cup \cdots \cup V_{ik(i)}$ of homeomorphic copies of V_i. (If $n_i = 1$, let $W_i = \varnothing$.) Define $f_0(x) = p_{ij}$ for $x \in V_{ij}$. Then $f_0 \in C(X_0, Z)$ and by the first part of the proof

$$C(X_0, Z)/f_0 C(X_0, Z) \cong \prod_i \prod_{j=1}^{k(i)} C(V_i, Z_{p_{ij}}).$$

Thus, there is a homomorphism of $C(X_0, Z)$ onto F whose kernel M_0 contains the prime-valued function f_0.

The proof of 4.5.4 shows more than is stated. This surplus information will be useful later.

COROLLARY 4.5.5. *Let* $\Pi_0 \in \mathcal{P}(M)$, *where* M *is a maximal ideal of* $C(X, Z)$. *Then there exist spaces* U_p *for each* $p \in \Pi_0$ *such that* $C(X, Z)/M$ *is a homomorphic image of the ring* $\prod_{p \in \Pi_0} C(U_p, Z_p)$.

Proof. This is clear if X is a *ZPC* space. Suppose that X is not *ZPC*. In the proof of 4.5.4, choose f initially so that $\pi(D(f)) = \Pi_0$. Then the argument given shows that $C(X, Z)/M$ is the homomorphic image of $\prod_{p \in \Pi_0} C(U_p, Z_p)$, where U_p is the topological sum of all those spaces V_{ij} for which $p_{ij} = p$.

5. **Cardinality of residue class fields.** The object of this section is to prove two results concerning the cardinality of the residue class fields $C(X, Z)/M$, where M is a maximal ideal of $C(X, Z)$. In view of 3.2.3, we may as well assume that X is not a *ZPC* space and that $M \in \mathfrak{Z}(X)$.

5.1. We first establish a general result.

THEOREM 5.1.1. *If* $M \in \mathfrak{Z}(X)$, *then* $C(X, Z)/M$ *is uncountable.*

Proof. By 4.5.4, we may assume that M contains a function f such that for all x, either $f(x) = 1$ or $f(x)$ is a prime. Let $p_1, p_2, \cdots$ be an enumeration in order of magnitude of those distinct primes p such that $f(x) = p$ for some $x \in X$. Since $M \in \mathfrak{Z}(X)$, this sequence is infinite (see 3.3.4). Note that $p_j > j$ for $j = 1, 2, \cdots$. Let $V_0 = \{x \in X | f(x) = 1\}$, $V_j = \{x \in X | f(x) = p_j\}$. Then $\{V_j\}$ is an infinite partition of X. Suppose that $g_1, g_2, \cdots$ is any sequence of functions in $C(X, Z)$. We will show that there is an $h \in C(X, Z)$ such that $D(f) \cap D(h - g_i) \subseteq X \times \{p_1, \cdots, p_{i-1}\}$. Since $M \in \mathfrak{Z}(X)$, this implies that $D(h - g_i) \cap D_i = \varnothing$ for some $D_i \in D(M)$, by 3.3.3. Therefore, $h - g_i \notin M$ by 2.3.6. This will complete the proof. For each $i > 0$, let $\{V_{ij}\}$ be a partition of V_i such that all of the functions $g_1, g_2, \cdots, g_i$ are constant on each of

the open-and-closed sets V_{ij}. Define h by the following conditions: $h(x) = 0$ for $x \in V_0$; $h(x) = n_{ij}$ for $x \in V_{ij}$, where n_{ij} is any integer which is not congruent modulo p_i to any of the numbers $g_1(x), \cdots, g_i(x)$, $x \in V_{ij}$ (which are independent of the choice of x by the definition of V_{ij}). This is possible since $p_i > i$. It then follows that p_i does not divide any of the numbers $h(x) - g_1(x)$, $\cdots, h(x) - g_i(x)$, if $x \in V_i$. Thus, $D(f) \cap D(h - g_i) \subseteq X \times \{p_1, \cdots, p_{i-1}\}$, as required.

COROLLARY 5.1.2. *If $M \in \mathfrak{B}(X)$, then the transcendence degree of $C(X, Z)/M$ over its prime field is uncountable.*

5.2. We now establish the existence of residue class fields of arbitrarily large cardinality. The proof is a slight modification of the proof of the corresponding result for rings of real-valued continuous functions given in [2, p. 166].

THEOREM. *Let X be a discrete space of infinite cardinality α. Then $C(X, Z)$ contains a maximal ideal M such that the cardinality of $C(X, Z)/M$ is greater than α.*

Proof. Let $\{F_x | x \in X\}$ be an indexing of the finite nonempty subsets of X by elements of X. For $y \in X$, define $V_x = \{y \in X | x \in F_y\}$. That is, $y \in V_x$ if and only if $x \in F_y$. In particular, if $F_y = \{x_1, \cdots, x_j\}$, then $y \in V_{x_1} \cap \cdots \cap V_{x_j}$. Thus, there is an ultrafilter $\mathfrak{U}$ of $\mathfrak{B}(X)$ containing all V_x. For each $x \in X$, choose $p_x \in \Pi$ such that p_x exceeds the number of elements in F_x. Define $f \in C(X, Z)$ by $f(x) = p_x$ for all x. Let $M = M(f, \mathfrak{U})$ as in 4.3.2. We wish to prove that the cardinality of $C(X, Z)/M$ exceeds α. Suppose $E \subset C(X, Z)$ has cardinality at most α. Index E by X, allowing repetition: $E = \{g_y | y \in X\}$. We will define $h \in C(X, Z)$ so that $h - g_y \notin M$ for all $y \in X$. This will complete the proof. For $x \in X$, let $h(x)$ be any integer which is distinct modulo p_x from all of the numbers $\{g_y(x) | y \in F_x\}$. That is, if $y \in F_x$, then p_x does not divide $h(x) - g_y(x)$. This is possible since F_x contains fewer than p_x elements. It follows from this choice that for a fixed y, $f(x)$ does not divide $h(x) - g_y(x)$ if $x \in V_y$. That is,

$$D(h - g_y) \cap (D(f) \cap (V_y \times \Pi)) = \varnothing.$$

Consequently, $h - g_y \notin M$.

6. Arithmetic properties of residue class fields.

6.1. DEFINITION. An *arithmetic sentence* is a formal expression

$$A = [Q](P(\lambda_1, \lambda_2, \cdots, \lambda_n) = 0),$$

where

(a) $P(\lambda_1, \lambda_2, \cdots, \lambda_n)$ is a formal polynomial, that is, a sum or difference of products of the variables $\lambda_1, \lambda_2, \cdots, \lambda_n$ and the individual constant 1,

(b) $[Q] = (Q_1\lambda_1)(Q_2\lambda_2) \cdots (Q_n\lambda_n)$, with each Q_i one of the quantifiers $\exists$ or $\forall$.

For example,

$$(\forall\lambda_1)(\exists\lambda_2)(\lambda_2\lambda_2 - \lambda_2 - \lambda_2 + 1 + 1 = 0)$$

is an arithmetical sentence. Since we will only be concerned with arithmetical sentences as statements about commutative rings with identity, it can be assumed that the formal polynomials involved satisfy the laws of commutative rings. Thus, for instance, the sentences $(\forall\lambda_1)(\exists\lambda_2)(-\lambda_1+\lambda_2\lambda_1\lambda_2+\lambda_1-1=0)$ and $(\forall\lambda_1)(\exists\lambda_2)(\lambda_2\lambda_2\lambda_1-1=0)$ will be considered as equivalent. Moreover, it is convenient to introduce exponents as abbreviations for repeated products and numerical constant coefficients as abbreviations for repeated summands. Thus, instead of $\lambda_1\lambda_1\lambda_1$, $\lambda_1+\lambda_1+\lambda_1+\lambda_1$, $1+1$, we write λ_1^3, $4\lambda_1$, 2 respectively.

NOTATION. Let $I = \langle i_1, i_2, \cdots, i_r \rangle$ be an ordered subset (possibly empty) of $\langle 1, 2, \cdots, n \rangle$. Write λ_I for $\langle \lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_r} \rangle$, $\exists\lambda_I$ for $(\exists\lambda_{i_1})(\exists\lambda_{i_2}) \cdots (\exists\lambda_{i_r})$ and $\forall\lambda_I$ for $(\forall\lambda_{i_1})(\forall\lambda_{i_2}) \cdots (\forall\lambda_{i_r})$.

6.2. We will be concerned with the interpretation of arithmetical sentences in commutative rings, particularly fields. With such an interpretation, formal polynomials take on their usual meaning as operations defined on the ring. Arithmetical sentences become statements of arithmetical properties of the ring which may either be true or false in any particular model. It is possible to give a formal definition of the interpretation of arithmetical sentences and in particular of the validity of arithmetical sentences in a given ring (see [9] for example). We prefer however to proceed informally.

DEFINITION. Let R be a commutative ring with an identity. Then the arithmetical sentence

$$(\forall\lambda_{I_1})(\exists\lambda_{K_1}) \cdots (\forall\lambda_{I_r})(\exists\lambda_{K_r})(P(\lambda_{I_1}, \lambda_{K_1}, \cdots, \lambda_{I_r}, \lambda_{K_r}) = 0)$$

is *valid* in R if there exist functions

$$\xi_j(\lambda_{I_1}, \cdots, \lambda_{I_j}): R^{I_1} \times \cdots \times R^{I_j} \to R^{K_j},$$

such that for every $\langle \lambda_{I_1}, \cdots, \lambda_{I_r} \rangle \in R^{I_1} \times \cdots \times R^{I_r}$, we have

$$P(\lambda_{I_1}, \xi_1(\lambda_{I_1}), \cdots, \lambda_{I_r}, \xi_r(\lambda_{I_1}, \cdots, \lambda_{I_r})) = 0.$$

It is not hard to show (using the axiom of choice) that this definition of validity agrees with the usual formal concept of validity.

6.3. The notion of an arithmetical sentence is very general. In fact, any first order statement about a field can be expressed as an arithmetical sentence. That is, given any first order sentence S in the formalism of ring theory, there is an arithmetical sentence A such that for any field F, S is valid in F if and only if A is valid in F. This assertion is an easy consequence of the following observation.

LEMMA. *It is possible to deduce from the axioms of field theory*

(a) $\sim(P(\lambda_1, \lambda_2, \cdots, \lambda_n) = 0) \equiv (\exists \mu)(\mu P(\lambda_1, \lambda_2, \cdots, \lambda_n) - 1 = 0)$,

(b) $(P_1(\lambda_1, \lambda_2, \cdots, \lambda_n) = 0) \vee \cdots \vee (P_r(\lambda_1, \lambda_2, \cdots, \lambda_n) = 0)$

$$\equiv P_1(\lambda_1, \lambda_2, \cdots, \lambda_n) \cdots P_r(\lambda_1, \lambda_2, \cdots, \lambda_n) = 0.$$

6.4. We are now ready to state and prove the main result of this section.

THEOREM 6.4.1. *Let A be an arithmetical sentence. Let $\Pi_0 = \{p \in \Pi \mid A$ is valid in $Z_p\}$. Suppose that X is any space and M is a maximal ideal of $C(X, Z)$. Then A is valid in $C(X, Z)/M$ if and only if $\Pi_0 \in \mathcal{O}(M)$.*

The proof of this result uses several simple lemmas, the first two of which are well known (see [6] and the references given there).

LEMMA 6.4.2. *Let A be valid in the ring R. Suppose that $\overline{R}$ is a homomorphic image of R. Then A is valid in $\overline{R}$.*

LEMMA 6.4.3. *If A is valid in each of the rings R_i, then A is valid in the direct product $\prod_i R_i$.*

LEMMA 6.4.4. *If A is valid in Z_p, then A is valid in $C(V, Z_p)$ for every space V.*

Proof. Let $A = (\forall \lambda_{I_1})(\exists \lambda_{K_1}) \cdots (\forall \lambda_{I_r})(\exists \lambda_{K_r})(P(\lambda_{I_1}, \lambda_{K_1}, \cdots, \lambda_{I_r}, \lambda_{K_r}) = 0)$. Since A is valid in Z_p, there exists $\xi_j(\lambda_{I_1}, \cdots, \lambda_{I_j}) : Z_p^{I_1} \times \cdots \times Z_p^{I_j} \to Z_p^{K_j}$ such that $P(\lambda_{I_1}, \xi_1(\lambda_{I_1}), \cdots, \lambda_{I_r}, \xi_r(\lambda_{I_1}, \cdots, \lambda_{I_r}))$ is identically zero in Z_p. Define $\bar{\xi}_j(f_{I_1}, \cdots, f_{I_j}) : C(V, Z_p)^{I_1} \times \cdots \times C(V, Z_p)^{I_j} \to C(V, Z_p)^{K_j}$ by

$$\bar{\xi}_j(f_{I_1}, \cdots, f_{I_j})(x) = \xi_j(f_{I_1}(x), \cdots, f_{I_j}(x)).$$

The important thing to notice is that since Z_p is discrete, $\bar{\xi}_j$ actually does map into $C(V, Z_p)^{K_j}$. Now clearly $P(f_{I_1}, \bar{\xi}_1(f_{I_1}), \cdots, f_{I_r}, \bar{\xi}_r(f_{I_1}, \cdots, f_{I_r}))$ is identically zero. Thus, by definition, A is valid in $C(V, Z_p)$.

Proof of 6.4.1. Assume that $\Pi_0 \in \mathcal{O}(M)$. By 4.5.5, $C(X, Z)/M$ is a homomorphic image of $\prod_{p \in \Pi_0} C(V_p, Z_p)$, for suitable spaces V_p. Consequently, A is valid in $C(X, Z)/M$ by 6.4.4, 6.4.3 and 6.4.2. Now suppose that $\Pi_0 \notin \mathcal{O}(M)$. Then by 4.1.4, $\Pi_0^c \in \mathcal{O}(M)$. Moreover, $\sim A$ is valid in Z_p for all $p \in \Pi_0^c$. Thus, by 6.3 and the first part of the proof, $\sim A$ is valid in $C(X, Z)/M$. Thus, A is not valid in $C(X, Z)/M$.

COROLLARY 6.4.5. *Let X be an arbitrary space. Suppose that A is an arithmetical sentence. Then A is valid in $C(X, Z)/M$ for every $M \in \mathcal{B}(X)$ if and only if A is valid in Z_p for almost all primes p.*

COROLLARY 6.4.6. *Let X be a space which is not Z-pseudocompact. Let $A_1, A_2, \cdots$ be a sequence of arithmetical sentences. Define $\Pi_j = \{p \in \Pi \mid A_j$ is valid in $Z_p\}$. Then there exists a maximal ideal M belonging to $\mathcal{B}(X)$ such that each of the sentences $A_1, A_2, \cdots$ is valid in $C(X, Z)/M$ if and only if every intersection $\Pi_1 \cap \Pi_2 \cap \cdots \cap \Pi_n$ contains infinitely many primes.*

6.5. We will use the results of 6.4.5 to obtain some properties which are common to all residue class fields of $C(X, Z)$.

THEOREM 6.5.1. *If M is a maximal ideal of $C(X, Z)$, then the field $C(X, Z)/M$ is quasi-algebraically closed. That is, every homogeneous form of degree d in n variables with $n > d$ has a nontrivial solution in $C(X, Z)/M$.*

Proof. Let $P(\mu_I, \lambda_1, \cdots, \lambda_n) = \sum_{i \in I} \mu_i M_i(\lambda_1, \cdots, \lambda_n)$, where $M_i(\lambda_1, \cdots, \lambda_n)$ ranges over all monomials of degree d in $\lambda_1, \cdots, \lambda_n$. Let $P_1(\mu_I, \lambda_1, \cdots, \lambda_n) = P(\mu_I, 1, \lambda_2, \cdots, \lambda_n) P(\mu_I, \lambda_1, 1, \cdots, \lambda_n) \cdots P(\mu_I, \lambda_1, \lambda_2, \cdots, 1)$. Define A_{nd} to be the arithmetical sentence

$$(\forall \mu_I)(\exists \lambda_K)(P_1(\mu_I, \lambda_K) = 0), \quad K = \langle 1, \cdots, n \rangle.$$

The statement that a field F is quasi-algebraically closed is equivalent to the statement that A_{nd} is valid in F whenever $n > d$. Since every finite field is quasi-algebraically closed (see [1]), it follows from 6.4.5 that $C(X, Z)/M$ is quasi-algebraically closed for every maximal ideal M.

COROLLARY 6.5.2. *Every element of $C(X, Z)/M$ can be represented as a sum of two squares.*

Proof. If $C(X, Z)/M \cong Z_2$, this is trivial. Otherwise, the assertion follows from 3.3.2, 6.5.1 and the theory of quadratic forms (see [5, p. 41]).

Although the residue class fields of $C(X, Z)$ are quasi-algebraically closed, they can never be algebraically closed. This observation is due to Professor J. R. Isbell.

THEOREM 6.5.3 (ISBELL). *If M is a maximal ideal in $C(X, Z)$, then $C(X, Z)/M$ is not algebraically closed. In fact, unless $C(X, Z)/M \cong Z_2$, there is an element λ in $C(X, Z)/M$ which has no square root.*

Proof. The sentence $(\forall \lambda)(\exists \mu)(\mu^2 - \lambda = 0)$ is not valid in Z_p for each prime $p > 2$. Thus, if $\{2\}^c \in \mathcal{P}(M)$, the sentence is not valid in $C(X, Z)/M$ by 6.4.1.

6.6. We will now consider the set of all algebraic elements of the field $C(X, Z)/M$.

DEFINITION 6.6.1. Let M be a maximal ideal in $C(X, Z)$. Denote by $\mathfrak{a}(M)$ the set of all elements of $C(X, Z)/M$ which are algebraic over the prime field.

Thus, if $M \in \mathfrak{P}(X)$, then $\mathfrak{a}(M) = C(X, Z)/M \cong Z_p$ for some prime p. On the other hand, if $M \in \mathfrak{Z}(X)$, then $\mathfrak{a}(M)$ is a proper subfield of $C(X, Z)/M$ by 5.1.2. We will prove that $\mathfrak{a}(M)$ is uniquely determined by the ultrafilter $\mathcal{P}(M)$.

THEOREM 6.6.2. *Let M_i be a maximal ideal in $C(X_i, Z)$ for $i = 1, 2$. Suppose that $\mathcal{P}(M_1) = \mathcal{P}(M_2)$. Then $\mathfrak{a}(M_1) \cong \mathfrak{a}(M_2)$.*

Proof. By 3.3.3, we may assume that M_1 and M_2 are in $\mathfrak{Z}(X)$. Then using 6.4.1 the theorem is a consequence of the following algebraic result.

LEMMA 6.6.3. *Let F_1 and F_2 be subfields of the algebraic closure Ω of the rational field Q. Suppose that $\{P \in Q[\lambda] \mid P$ has a root in $F_1\} = \{P \in Q[\lambda] \mid P$ has a root in $F_2\}$. Then $F_1 \cong F_2$.*

Proof. It suffices to show that there is an isomorphism ϕ of F_1 into F_2. For then by symmetry there is an isomorphism ψ of F_2 into F_1. The isomorphism $\phi\psi$ of F_2 into itself is necessarily onto. Indeed if E is a finite normal subfield of Ω, then $\phi\psi(E \cap F_2) \subseteq E \cap F_2$ and since $[\phi\psi(E \cap F_2) : Q] = [E \cap F_2 : Q]$, equality must hold. However every element of F_2 is in a finite normal subfield of Ω. Thus $\phi\psi$ is onto and consequently so is ϕ. Let $\mathfrak{G}$ be the group of all automorphisms of Ω. Then, as is well known, $\mathfrak{G}$ is a compact topological group with the topology defined by taking for a neighborhood basis of the identity those subgroups $\mathfrak{N}_E$ consisting of all automorphisms which leave fixed all the elements of a finite extension E of Q. If E is a finite subfield of F_1, let $\mathfrak{K}_E = \{\phi \in \mathfrak{G} \mid \phi(E) \subseteq F_2\}$. First note that $\mathfrak{K}_E$ is closed in $\mathfrak{G}$. In fact, $\mathfrak{K}_E \mathfrak{N}_E = \mathfrak{K}_E$ so that the complement of $\mathfrak{K}_E$ is a union of cosets of the open subgroup $\mathfrak{N}_E$. Also, $\mathfrak{K}_E$ is not empty. Indeed, $E = Q(\alpha)$ for some $\alpha \in F_1$. Let P be the field polynomial of α over Q. Then by the hypothesis of the lemma, there is an $\alpha' \in F_2$ such that $P(\alpha') = 0$. Consequently, there is an isomorphism ϕ of E into F_2 and this can be extended to an automorphism of Ω which, by its definition, belongs to $\mathfrak{K}_E$. Since the finite subfields of F_1 are directed by inclusion, it follows that the closed sets $\mathfrak{K}_E$ have the finite intersection property. The compactness of $\mathfrak{G}$ implies that there is an automorphism ϕ of Ω such that $\phi(E) \subseteq F_2$ for all subfields E of F_1 which are finite over Q. Thus, $\phi(F_1) \subseteq F_2$. This completes the proof of 6.6.3 and 6.6.2.

6.7. The fields $\mathfrak{a}(M)$ cannot be entirely arbitrary. This is shown, for example, by the following result.

THEOREM 6.7.1. *If $M \in \mathfrak{Z}(X)$, then the field $\mathfrak{a}(M)$ has infinite degree over its prime field.*

Proof. If p is any prime, and if m and n are integers, then at least one of the quantities m, n or mn is a quadratic residue modulo p. That is, the sentence $(\exists \lambda)((\lambda^2 - m)(\lambda^2 - m)(\lambda^2 - mn) = 0)$ is valid in Z_p for every p. Consequently, by 6.4.5 it is valid in $C(X, Z)/M$ for every M. Thus, for any two elements $\bar{m}$, $\bar{n}$ of the prime field of $C(X, Z)/M$, at least one of $\bar{m}$, $\bar{n}$ or $\bar{m}\bar{n}$ has a square root in $\mathfrak{a}(M)$. If $M \in \mathfrak{Z}(X)$, various choices of $\bar{m}$, $\bar{n}$ will give an infinite linearly independent set over the prime field of $\mathfrak{a}(M)$.

The question of which fields can be realized in the form $\mathfrak{a}(M)$ seems to be difficult. One might hope that all such fields are normal. However this is not the case.

EXAMPLE 6.7.2. *If X is not a ZPC space, then there is a maximal ideal $M \in \mathfrak{Z}(X)$ such that $\mathfrak{a}(M)$ is not normal over its prime field.*

Proof. It is well known (see, for example, $[7, \text{p. } 116]$) that the congruence $\lambda^3 + 2 \pmod{p}$ has exactly one solution if p is a prime of the form $6m - 1$, $m \geq 1$. Let Π_0 be the set of all such primes. Then the arithmetic sentence $A_1 = (\exists \lambda)(\lambda^3 - 2 = 0)$ is valid in Z_p for all $p \in \Pi_0$. On the other hand, the sentence

$$A_2 = (\exists \lambda_1)(\exists \lambda_2)(\exists \lambda_3)(\forall \mu_1)(\forall \mu_2)(\forall \mu_3)(\mu_1(\lambda_1 + \lambda_2 + \lambda_3)$$

$$+ \mu_2(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1) + \mu_3(\lambda_1 \lambda_2 \lambda_3 - 2) = 0)$$

fails to be valid in each Z_p with $p \in \Pi_0$. Indeed A_2 is valid in a field F if and only if the polynomial $\lambda^3 - 2$ splits completely in F. Let M be a maximal ideal in $\mathfrak{Z}(X)$ such that $\Pi_0 \in \mathcal{P}(M)$. Such an ideal exists by 4.2.3. By 6.4.1, A_1 is valid in $C(X, Z)/M$, but A_2 is not valid in $C(X, Z)/M$. Thus $\mathfrak{a}(M)$ contains precisely one root of the polynomial $\lambda^3 - 2$. Therefore $\mathfrak{a}(M)$ is not normal.

We conclude this section by showing that the field of all algebraic numbers can be realized in the form $\mathfrak{a}(M)$.

EXAMPLE 6.7.3. If X is not a ZPC space, then there exists a maximal ideal $M \in \mathfrak{Z}(X)$ such that $\mathfrak{a}(M)$ is algebraically closed.

Proof. Let $P_1, P_2, \cdots$ be an enumeration of all monic polynomials in the indeterminate λ with integral coefficients. Let $A_i = (\exists \lambda)(P_i(\lambda) = 0)$, and let $\Pi_i = \{p \in \Pi \mid A_i \text{ is valid in } Z_p\}$. In view of 6.4.6, it is sufficient to prove the following two lemmas.

LEMMA 6.7.4. *For any n, $\Pi_1 \cap \Pi_2 \cap \cdots \cap \Pi_n$ is infinite.*

LEMMA 6.7.5. *If F is a field of characteristic zero in which $A_1, A_2, \cdots$ are all valid, then F contains the algebraic closure of its prime field.*

Proof of 6.7.4. Let K be the subfield of the complex numbers which is generated by the roots of $P_1, \cdots, P_n$. Then K is a finite, normal extension of the rational field Q. Let θ be an algebraic integer such that $K = Q(\theta)$. Let P be the field polynomial of θ over Q. Then P is a monic, irreducible polynomial with coefficients in Z. Let d be the discriminant of P. Since P has roots modulo an infinity of primes, it is sufficient to show that if P has a root modulo p and p does not divide d, then each of the polynomials $P_1, \cdots, P_n$ has a root modulo p. If p does not divide d, the roots of P modulo p are distinct. If also P has a root modulo p, Hensel's Lemma implies that there is a p-adic integer α such that $P(\alpha) = 0$. Consequently, the field $Q(\alpha)$ is isomorphic to $Q(\theta)$. Therefore $P_1, P_2, \cdots, P_n$ split in $Q(\alpha)$. In particular, there exist p-adic integers $\beta_1, \cdots, \beta_n$ such that $P_1(\beta_1) = \cdots = P_n(\beta_n) = 0$. It follows that each P_i has a root modulo p.

Proof of 6.7.5. If $A_1, A_2, \cdots$ are all valid in F, then clearly every poly-

nomial of positive degree with rational coefficients has a root in F. Thus, 6.7.5 is a special case of 6.6.3.

7. Unsolved problems.

7.1. How close is the relation between the algebraic structure of $C(X, Z)/M$ and the filter $\mathcal{P}(M)$? In particular, does $C(X, Z)/M \cong C(X, Z)/N$ imply $\mathcal{P}(M) = \mathcal{P}(N)$? Note by cardinality considerations that there exist ideals M and N such that $\mathcal{A}(M) \cong \mathcal{A}(N)$ and $\mathcal{P}(M) \neq \mathcal{P}(N)$.

7.2. What relations exist between the ultrafilters $\mathcal{P}(M)$ and $\mathcal{U}(M)$? It is clear, for example, that if $\mathcal{U}(M)$ is fixed, then the same is true of $\mathcal{P}(M)$.

7.3. For maximal ideals M and N of $C(X, Z)$, do the equalities $\mathcal{P}(M) = \mathcal{P}(N)$ and $\mathcal{U}(M) = \mathcal{U}(N)$ imply the equality $M = N$?

For convenience in stating the next three problems, let us say that a field F is a *residue class field* if there is a space X and a maximal ideal M in $C(X, Z)$ such that $F \cong C(X, Z)/M$.

7.4. What are the possible cardinalities of residue class fields? If F is a residue class field, is it true that $|F|^{\aleph_0} = |F|$?

7.5. If F_1 and F_2 are residue class fields which are contained in a field F, is there a residue class field containing a field isomorphic to F?

7.6. If F is a residue class field, is there a discrete space X and a maximal ideal M in $C(X, Z)$ such that $F \cong C(X, Z)/M$? Note that the corresponding problem for $\mathcal{A}(M)$ has an affirmative solution by 6.6.2 and 4.2.2.

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